Ratio of Quark Masses in Duality Theories

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Abstract

We consider $N = 2$ $SU(2)$ Seiberg–Witten duality theory for models with $N_f = 2$ and $N_f = 3$ quark flavors. We investigate arbitrary large bare mass ratios between the two or three quarks at the singular points. For $N_f = 2$ we explore large bare mass ratios corresponding to a singularity in the strong coupling region. For $N_f = 3$ we determine the location of both strong and weak coupling singularities that produce specific large bare mass ratios.

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1 N = 2 SU(2) QCD with Bare Mass Quarks

Understanding of the vacuum structure of $N = 2$ supersymmetric gauge theories in four spacetime dimensions has progressed significantly in recent years. For example, the moduli space of $N = 2$ supersymmetric $SU(2)$ QCD is now known to be the complex u–plane with its singularities. Physically, u is the vacuum expectation value of the square of a complex scalar field, ϕ, in the adjoint representation of $SU(2)$, $u = \langle \text{Tr}\,\phi^2 \rangle$. The u–plane singularities are described by their monodromy matrices [1, 2]. To every value of u there corresponds a genus one Riemann surface that can be represented by a curve of the form

$$y^2 = F(x, u)\,, \tag{1.1}$$

where F is a cubic polynomial in x,

$$F = x^3 + \beta x^2 + \gamma x + \delta\,. \tag{1.2}$$

Thus, (1.1) yields a family of elliptic curves over the parameter space of u.

Associated with any polynomial F is its discriminant Δ defined by

$$\Delta = \prod_{i<j}(e_i - e_j)^2, \tag{1.3}$$

where the e_i are the roots of F. The branch points of the $N = 2$ family of curves $y^2 = F(x, u)$ overlap at the locations where the discriminant Δ is zero. In other words, the zeros of Δ specify the locations of the singularities in u parameter space. At the singularities, certain magnetic monopoles or dyons become massless. For a cubic polynomial F (1.2), the discriminant (1.3) can be expressed as

$$\Delta = -27\delta^2 + 18\beta\gamma\delta + \beta^2\gamma^2 - 4\beta^3\delta - 4\gamma^3\,. \tag{1.4}$$

In this letter, we examine the relationship between bare mass ratios of quark flavors and the location of the singularities. While the bare mass ratios are, indeed, free parameters of the theory, we show that the discriminant has predictive capabilities with regard to bare quark mass ratios *at the singularities*. We investigate this for both the two flavor and the three flavor cases.

2 The Two Quark Model

Consider first the $N = 2$, $N_f = 2$ Seiberg–Witten $SU(2)$ model, with related non–zero bare masses denoted m_a and m_b, where $m_a \geq m_b$. The family of curves of the modular space can be parametized by

$$y^2 = (x^2 - t^2)(x - u) + 2m^2 tx - 2M^2 t^2\,, \tag{2.1}$$

where the square of the energy scale of the theory is $t \equiv \frac{1}{8}\Lambda^2$. We have also used $m^2 \equiv m_a m_b$ and $M^2 \equiv \frac{1}{2}(m_a^2 + m_b^2)$. This equation is derived by Seiberg and Witten on the basis of conservation laws and appropriate boundary conditions [1, 2]. In (2.1), x, u, and t have mass dimension 2, while m, M, and N all have mass dimension 1. The mass dimension is one–half of the $U(1)_R$ charge of [1, 2].

Let us examine the possible mass hierarchy between the bare masses m_a and m_b. When the masses are equal, the discriminant of (2.1) is

$$\Delta = 4t^2[(u+t)^2 - 8m^2 t](u - t - m^2)^2 \,. \tag{2.2}$$

When the masses are unequal, let

$$M^2 \;=\; m^2 + (M^2 - m^2) \equiv m^2 + 2D^2 \tag{2.3}$$

and

$$M^2 \;=\; -m^2 + (M^2 + m^2) \equiv -m^2 + 2N^2. \tag{2.4}$$

In other words,

$$D^2 \;=\; \tfrac{1}{2}(M^2 - m^2) = \tfrac{1}{4}(m_a - m_b)^2 \,, \tag{2.5}$$

and

$$N^2 \;=\; \tfrac{1}{2}(M^2 + m^2) = \tfrac{1}{4}(m_a + m_b)^2 \,. \tag{2.6}$$

This gives

$$\Delta = 4t^2 \left[(u+t)^2 - 8m^2 t\right] (u - t - m^2)^2 + \Delta_{BNDY} \,, \tag{2.7}$$

where

$$\Delta_{BNDY} = -144 D^2 t^2 \left[(3t^2 - tu)N^2 + tuD^2 - t^2 u + \tfrac{1}{9}u^3\right] \,. \tag{2.8}$$

The condition $D^2 = 0$ yields, by definition, the equal mass case $m_a = m_b$; $D^2 > 0$ implies $m_a > m_b$.

We kept the form of the first term of Δ in (2.7) similar to that in (2.2) so we can decide on the region in the u space that we wish to focus on. At or near the singularities in the u–plane,

$$\Delta_{BNDY} = 0 \tag{2.9}$$

can be viewed as a boundary constraint when m_a and m_b are inequivalent. For $D^2 \neq 0$, (2.9) implies

$$(3t^2 - tu)N^2 + tuD^2 - t^2 u + \tfrac{1}{9}u^3 = 0 \,. \tag{2.10}$$

The three distinct roots of Δ in (2.2) (and Δ in (2.7) when $\Delta_{BNDY} = 0$) are $u = u_o \equiv t + m^2$, $u = u_+ \equiv -t + \sqrt{8m^2 t}$, and $u = u_- \equiv -t - \sqrt{8m^2 t}$. Consider the singular region around the double zero of Δ, $u_o \equiv t + m^2$. At this singularity, we find

$$m^2 = u - t = N^2 - D^2. \tag{2.11}$$

Together (2.11) and (2.10) yield,

$$N^2 = u\left(\frac{u}{3t} - \frac{1}{27}\frac{u^2}{t^2}\right) \tag{2.12}$$

and

$$D^2 = t\left(1 - \frac{u}{t} + \frac{u^2}{3t^2} - \frac{1}{27}\frac{u^3}{t^3}\right). \tag{2.13}$$

For

$$\frac{u}{t} = 1 + \epsilon, \tag{2.14}$$

where ϵ is regarded as small and positive, we obtain

$$N^2 = t\frac{8}{27}\left(1 + \frac{15}{8}\epsilon\right), \tag{2.15}$$

and

$$D^2 = t\frac{8}{27}\left(1 - \frac{3}{2}\epsilon\right). \tag{2.16}$$

Expressed in terms of N and D, the mass ratio is

$$\frac{m_a}{m_b} = \frac{N + D}{N - D}. \tag{2.17}$$

From Eqs. (2.15-2.17) we find

$$\frac{m_a}{m_b} = \frac{N + D}{N - D} = \frac{32}{27\epsilon} \tag{2.18}$$

at the double zero singularity, $u = t + m^2$. Since ϵ can be an arbitrarily small number as $u \to t$, the mass ratio can be arbitrarily large [3].

In concluding our study of the two mass case, we comment that in [4] Hanany and Oz started with a hyperelliptic modular curve

$$y^2 = (x^2 - u + t)^2 - 64t^2(x + m_a)(x + m_b), \tag{2.19}$$

and obtained the same Δ as in (2.2) for equal masses. For classical Lie groups, the Seiberg–Witten curves may always be expressed as hyperelliptic curves. $SU(2)$ is the only classical group that also allows the corresponding Seiberg–Witten curve to take elliptic form [5].

3 The Three Quark Model

Now consider the $N = 2$, $N_f = 3$ Seiberg–Witten $SU(2)$ model with related non–zero bare masses, m_a, m_b, and m_c, where $m_a \geq m_b \geq m_c$. For three quarks, the family of curves equation is [2]

$$y^2 = x^2(x - u) - t^2(x - u)^2 - 3M^2t^2(x - u) + 2m^3tx - 3P^4t^2 \,. \tag{3.1}$$

Here we have defined $t \equiv \Lambda/8$, $m^3 \equiv m_a m_b m_c$, $M^2 \equiv (m_a^2 + m_b^2 + m_c^2)/3$, and $P^4 \equiv (m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2)/3$. Note that x and u have mass dimension 2, while t, m, M, and P all have mass dimension 1. Let us also define for later use the variables G and H via

$$M^2 \;=\; m^2 + (M^2 - m^2) = m^2 + G^2 \,, \tag{3.2}$$

and

$$P^4 \;=\; m^4 + (P^4 - m^4) = m^4 + H^4. \tag{3.3}$$

In the case of three equivalent bare masses, $m_a = m_b = m_c$, we reach the limits $m = M = P$ and $G = H = 0$.

Eq. (3.1) can be rewritten into the polynomial form of (1.2), with

$$\beta \;=\; -t^2 - u, \tag{3.4}$$
$$\gamma \;=\; 2t^2 u + 2m^3 t - 3M^2 t^2, \tag{3.5}$$
$$\delta \;=\; -t^2 u^2 + 3M^2 t^2 u - 3P^4 t^2 \,. \tag{3.6}$$

From (1.4), we find the corresponding discriminant to be

$$
\begin{aligned}
\Delta \;=\; & t^2 \Big[-32m^9 t - 243 P^8 t^2 - 6P^4(t^2 - 2u)(27M^2 t^2 + 2t^4 - 8t^2 u - u^2) \\
& + (12M^2 + t^2 - 4u)(-3M^2 t^2 + u^2)^2 + 4m^6(36M^2 t^2 + t^4 - 22t^2 u + u^2) \\
& - 4m^3 t \left(54M^4 t^2 - 27P^4(t^2 + u) + u(-2t^4 + 11t^2 u - 11u^2) + \right. \\
& \left. 3M^2(t^4 - 13t^2 u + 10u^2) \right) \Big]
\end{aligned}
\tag{3.7}
$$

While the variables m, M, and N simplify the form of the discriminant (3.7), the alternative set given above, m, G, and H, are more useful for our mass ratio study. In the language of G and H, the discriminant separates into four components:

$$\Delta = \Delta(m, u, t) + \Delta(m, G, u, t) + \Delta(m, H, u, t) + \Delta(m, G, H, u, t). \tag{3.8}$$

The first component,

$$
\begin{aligned}
\Delta(m, u, t) \;=\; & -t^2(m^2 + mt - u)^3 \times \\
& \left[(32m^3 t + 3m^2 t^2 + 3mt^3) + (t^2 - 12mt)u - 4u^2 \right]
\end{aligned}
\tag{3.9}
$$

contains only m, u, and t. The entire discriminant reduces to just this term for the equal mass case $m_a = m_b = m_c$. The second, third, and fourth terms,

$$
\begin{aligned}
\Delta(m, G, u, t) &= 3G^2 t^2 \Big[36 G^4 t^4 + 3G^2 t^2 (-24 m^3 t + 36 m^2 t^2 + t^4 + 4t^2 u + 8u^2) \\
&\quad + \Big(48 m^6 t^2 - 144 m^5 t^3 + 54 m^4 (t^4 + 2t^2 u) \\
&\quad - 2u^2 (t^4 - 4t^2 u - 2u^2) - 4m^3 (t^5 - 13 t^3 u + 10 t u^2) \\
&\quad + 6m^2 (t^6 - 4t^4 u - 8t^2 u^2) \Big) \Big]
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\Delta(m, H, u, t) &= -H^4 t^2 \Big[243 H^4 t^2 + 6 \Big(81 m^4 t^2 + (27 m^2 t^2 + 2t^4 - 8t^2 u - u^2) \times \\
&\quad (t^2 - 2u) - 18 m^3 t (t^2 + u) \Big) \Big]
\end{aligned}
\tag{3.11}
$$

$$
\Delta(m, G, H, u, t) = -162 G^2 H^4 t^4 (t^2 - 2u)
\tag{3.12}
$$

additionally contain, G, H, and both G and H, respectively.

Moving away from the equivalent mass point, we can still effectively keep

$$
\Delta = \Delta(m, u, t)
\tag{3.13}
$$

by separately enforcing an additional boundary constraint

$$
\Delta_{BNDY} \equiv \Delta(m, G, u, t) + \Delta(m, H, u, t) + \Delta(m, G, H, u, t) = 0 .
\tag{3.14}
$$

Imposition of the boundary constraint (3.14) allows us to solve for u at the singular points simply in terms of m and t. That is, the singularities are located at the values of u such that $\Delta(m, u, t) = 0$. There is a triple zero of (3.9) at

$$
u \equiv u_o = m^2 + mt
\tag{3.15}
$$

along with additional zeros at

$$
u \equiv u_+ = \frac{1}{8} \left(-12 mt + t^2 + \sqrt{t(8m + t)^3} \right) ,
\tag{3.16}
$$

and

$$
u \equiv u_- = \frac{1}{8} \left(-12 mt + t^2 - \sqrt{t(8m + t)^3} \right) .
\tag{3.17}
$$

In the $(\sqrt[3]{|m_a m_b m_c|} = m) << t$ limit,

$$
u_o \rightarrow mt = m\Lambda/8
\tag{3.18}
$$

$$
u_+ \rightarrow 2t^2 = \Lambda^2/32
\tag{3.19}
$$

$$
u_- \rightarrow -3mt = -3m\Lambda/8 .
\tag{3.20}
$$

Therefore for small m, we find $|u| \ll \Lambda^2$ in the regions near any of the three singularities. Since weak coupling corresponds to $|u| \gg \Lambda^2$, $m \ll t$ implies strong coupling

in the neighborhood of the singularities. Strong coupling also results when m and t are of the same magnitude. Only in the $m \gg t$ limit do the u singularities move into the weak coupling realm.

At each of the three distinct zeros of (3.9), the direct dependence of our boundary discriminant Δ_{BNDY} on u is removed by making the appropriate root substitution, (3.15), (3.16), or (3.17). We can always scale t to unity, thereby effectively defining m, G, and H in units of t. Thus, at a given singularity, Δ_{BNDY} becomes a polynomial involving only m, G, and H,

$$\Delta_{BNDY} \equiv \Delta(m, G) + \Delta(m, H) + \Delta(m, G, H) = 0. \tag{3.21}$$

for $u = u_o$ or u_+ or u_-.

We can solve (3.21) for any variable from the set $\{m, G, H\}$ in terms of the other two. Recall however that, irregardless of (3.21), m, G, and H are not totally independent parameters, at least if they are to result in necessarily real and positive m_a^2, m_b^2, and m_c^2. Some (m, G, H) solutions to (3.21) may result in unacceptable (i.e., negative or complex) values of the bare mass–squares.

The three equations defining m, M and P may be combined to form a polynomial

$$x^3 - 3M^2 x^2 + 3P^4 x - m^6 = 0, \tag{3.22}$$

where $x \equiv m_a^2$ or m_b^2 or m_c^2. Equivalently,

$$x^3 - 3(m^2 + G^2)x^2 + 3(m^4 + H^4)x - m^6 = 0. \tag{3.23}$$

Thus, the viable m, G, and H combinations are those such that *all three* roots of (3.23), corresponding to m_a^2, m_b^2, and m_c^2, are real and positive. One trivial constraint is $H, G \geq 0$. Further, we find that $H = 0$ is physically allowed only when $G = 0$ simultaneously, i.e., when all masses are equivalent. Specifically $H = 0$ and $G > 0$ implies that two mass–squares are negative, and likewise for $G = 0$ and $H > 0$.

One approach to generating a consistent set of masses $\{m_a, m_b, m_c\}$ for a given $u = u_o$ or u_+ or u_- singularity is to determine the general structure of m, G, and H solutions to $\Delta_{BNDY} = 0$. An alternate, albeit less general, method is to rewrite the boundary constraint (3.21) directly in terms of the three bare masses, $m_{i=a,b,c}$. (See Appendix A.) Following this, we can specify a ratio between the three masses,

$$m_a/m_a : m_b/m_a : m_c/m_a = 1 : b : c, \tag{3.24}$$

where $1 \geq b = m_b/m_a \geq c = m_c/m_a > 0$. Next we choose a singularity type $u = u_o$ (3.15), $u = u_+$ (3.16), or $u = u_-$ (3.17), and rewrite u in terms of m_a, m_b and m_c. We then substitute $b\,m_a$ and $c\,m_a$ for m_b and m_c in $\Delta_{BNDY} = 0$. Hence, we can determine the allowed values of m_a for the given mass ratio (3.24). Knowledge of m_a, m_b, and m_c and the singularity type specifies the location of the associated singularity.

After a mass ratio (3.24) is *chosen*, the boundary constraint appears for the u_o singularity as a polynomial of tenth order having at least four zero roots. Thus, the

non–trivial m_a solutions are the roots of a sixth order polynomial. At the u_+ and u_- singularities the boundary constraint appears as an eighth–order polynomial (with at least two zero roots), with some terms generically containing an extra factor of $\sqrt{1 + r\, m_a}$, where r is a numerical coefficient. Roots of the sixth and eight order polynomials can be found using programs such as Mathematica or Maple.

We followed the mass ratio approach to learn where a u singularity is consistent with a large bare mass ratio. We considered, for example, the location of the singularities when the bare mass ratio is the order of the physical top, charm, and up mass ratio, $m_a/m_a : m_b/m_a : m_c/m_c \approx 1 : 7 \times 10^{-3} : 3 \times 10^{-5}$. For this ratio, the u_o singularity provides a solution of $m_a \approx 1600\Lambda$, $u = u_o \approx 74\Lambda^2$, which is still in the strong coupling region. The u_+ and u_- singularities offer similar strong coupling solutions: $m_a \approx 1100\Lambda$ at $u = u_+ \approx +13\Lambda^2$ and $m_a \approx 820\Lambda$ at $u = u_- \approx -9.7\Lambda^2$, respectively. In Table I of Appendix B we also present examples of large bare mass ratios for u_o, wherein the m_a and u solutions are in the ranges $0.1\,\Lambda \lesssim m_a \lesssim 2000\Lambda$ and $0.1\,\Lambda^2 \lesssim u \lesssim 200\Lambda^2$.

Generic three quark bare mass ratios have u_o solutions with $m_a \gtrsim \Lambda$ and $|u| \gtrsim \Lambda^2$. For a given mass ratio, the u_+ and u_- singularities typically, but not always, offer legitimate m_a and u solutions of the same magnitude as those obtained from u_o. In particular, u_+ or u_- may sometimes lack a valid solution when $m_b \approx m_c$, and instead require that m_a become complex.

Both u_+ and u_- do, however, provide some additional classes of mass ratio solutions that u_o does not allow. (See Appendix B.) In all but one of these additional classes *very* fine tuning of m_a and u is required to produce a specific three quark mass ratio. The non–fine tuning exception is a u_+ class of *extremely*–weak coupling solutions with $m_a \gg \Lambda$ and $u \gg \Lambda^2$.

To conclude this section, we comment that the corresponding Hanany and Oz family of hyperelliptic curves for $N_c = 2$, $N_f = 3$ is,

$$y^2 = \left(x^2 - u + \Lambda(\frac{m_a + m_b + m_c}{8} + \frac{x}{4})\right)^2 - \Lambda^3 \prod_{i=a}^{c} (x + m_i) \,. \qquad (3.25)$$

Hanany and Oz have also given the curves for $N_c = 3$, $N_f = 3$,

$$y^2 = \left(x^3 - u_2 x - \frac{u_3}{3} + \frac{\Lambda^3}{4}\right)^2 - \Lambda^3 \prod_{i=a}^{c} (x + m_i) \,. \qquad (3.26)$$

(See Eqs. (5.5) and (4.13) of [4], respectively.)

4 Discussion

We have studied bare quark mass ratios at the singular points on the complex u–plane of Seiberg–Witten $N = 2$ supersymmetric $N_f = 2$ and $N_f = 3$ $SU(N_c = 2)$ theory. We have shown that large bare mass hierarchies at the singular points can

occur for both the two quark and three quark models. For $N_f = 2$ we found that demanding large bare mass ratios at singularities placed the singularities in the strong coupling region. In contrast, for $N_f = 3$ we determined the respective singularities could be located in either the strong or weak coupling regions.

We would emphasize that in general the bare masses are not the physical masses; only in the weak coupling limit do the bare masses become physical masses. Nonetheless, large bare quark mass hierarchies at the singularities of the $N = 2$ $SU(N_c = 2)$ parameter space for two or three quark flavors may suggest a possible explanation for the phenomenologically known three generation mass hierarchy. Such an explanation would need not depend on non–renormalizable terms in the superpotential. This explanation would require an extrapolation from the $N = 2$, $N_c = 2$ theory discussed herein to the $N = 1$, $N_c = 3$ case. This suggests that the $N = 2$, $N_c = 3$ case should be investigated as a next step. This is, however, beyond the scope of this letter and so we leave this for future research.

It is interesting to note that the Seiberg–Witten equation for the family of curves can be obtained from M–theory as shown by Witten [6]. Additional relevant information is available in Ennes, et al. [7] and the references cited therein. Witten studied the $N = 2$ supersymmetric gauge theories in four dimensions by formulating them as the quantum field theories derived from a configuration of various D–branes. He considered, for example, $N_c = N_f = 3$ (c is color, f is flavor) quantum field theory of two parallel five–branes connected by 3 four–branes, with 3 six–branes between them in Type IIA superstring theory on R^{10}, and reinterpreted this configuration in M–theory. World volumes of five–branes, four–branes, and six–branes are parametrized by the coordinates $x^0 x^1 x^2 x^3$ and $x^4 x^5$, $x^0 x^1 x^2 x^3$ and x^6, $x^0 x^1 x^2 x^3$ and $x^7 x^8 x^9$, respectively.

In M–theory, the above brane configuration can be reinterpreted as a configuration of a single five–brane with world volume $R^4 \times \sum$ where $\sum$ is the Seiberg–Witten curve. It yields the structure of the Coulomb branch of the $N = 2$ theory. The curve $\sum$ is given by an algebraic equation in (x, y) space where $x = x^4 + ix^5$ and $y = \exp[-(x^6 + ix^{10})/R]$. In terms of $\tilde{y} = y + B/2$, one obtains

$$\tilde{y}^2 = (B(x)^2/4) - \Lambda^3 \prod_{i=a}^{c} (x + m_i), \tag{4.1}$$

where $B(x) = e(x^3 + u_2 x + u_3)$. This is the hyperelliptic curve for $N_c = N_f = 3$ obtained by Hanany and Oz in (3.26).

5 Acknowledgments

A $N_c = 2$, $N_f = 3$ Discriminant Boundary Constraint As Function of Bare Masses, m_a, m_b, m_c and Complex Parameter u

$$
\begin{aligned}
\Delta_{BNDY} =\ & t^8[(m_a^4 + m_b^4 + m_c^4) - 2(m_a^2 m_b^2 + m_a^2 m_c^2 + 2m_b^2 m_c^2) + 3m_a^{4/3} m_b^{4/3} m_c^{4/3}] \\
& +t^7[-4(m_a^3 m_b m_c + m_a m_b^3 m_c + m_a m_b m_c^3) + 12m_a^{5/3} m_b^{5/3} m_c^{5/3}] \\
& +t^6[4(m_a^6 + m_b^6 + m_c^6) - 6(m_a^4 m_b^2 + m_a^2 m_b^4 + m_a^4 m_c^2 + m_b^4 m_c^2 + m_a^2 m_c^4 + m_b^2 m_c^4) \\
& \quad + 24m_a^2 m_b^2 m_c^2 - 4(m_a^4 + m_b^4 + m_c^4)u + 16(m_a^2 m_b^2 + m_a^2 m_c^2 + m_b^2 m_c^2)u \\
& \quad - 36m_a^{4/3} m_b^{4/3} m_c^{4/3}u - 2(m_a^2 + m_b^2 + m_c^2)u^2 + 6m_a^{2/3} m_b^{2/3} m_c^{2/3}u^2] \\
& +t^5[-24(m_a^5 m_b m_c + m_a m_b^5 m_c + m_a m_b m_c^5) - 12(m_a^3 m_b^3 m_c + m_a^3 m_b m_c^3 + m_a m_b^3 m_c^3) \\
& \quad + 108m_a^{7/3} m_b^{7/3} m_c^{7/3} + 52(m_a^3 m_b m_c + m_a m_b^3 m_c + m_a m_b m_c^3)u \\
& \quad - 156m_a^{5/3} m_b^{5/3} m_c^{5/3}u] \\
& +t^4[-27(m_a^4 m_b^4 + m_a^4 m_c^4 + m_b^4 m_c^4) + 99m_a^{8/3} m_b^{8/3} m_c^{8/3} \\
& \quad - 6(m_a^4 m_b^2 m_c^2 + m_a^2 m_b^4 m_c^2 + m_a^2 m_b^2 m_c^4) \\
& \quad + 36(m_a^4 m_b^2 + m_a^2 m_b^4 + m_a^4 m_c^2 + m_a^2 m_c^4 + m_b^4 m_c^2 + m_b^2 m_c^4)u \\
& \quad - 216m_a^2 m_b^2 m_c^2 u - 8(m_a^4 + m_b^4 + m_c^4)u^2 - 46(m_a^2 m_b^2 + m_a^2 m_c^2 + m_b^2 m_c^2)u^2 \\
& \quad + 162m_a^{4/3} m_b^{4/3} m_c^{4/3}u^2 + 8(m_a^2 + m_b^2 + m_c^2)u^3 - 24m_a^{2/3} m_b^{2/3} m_c^{2/3}u^3] \\
& +t^3[36(m_a^3 m_b^3 m_c + m_a^3 m_b m_c^3 + m_a m_b^3 m_c^3)u - 108m_a^{7/3} m_b^{7/3} m_c^{7/3}u \\
& \quad - 40(m_a^3 m_b m_c + m_a m_b^3 m_c + m_a m_b m_c^3)u^2 + 120m_a^{5/3} m_b^{5/3} m_c^{5/3}u^2] \\
& +t^2[-4(m_a^2 m_b^2 + m_a^2 m_c^2 + m_b^2 m_c^2)u^3 + 12m_a^{4/3} m_b^{4/3} m_c^{4/3}u^3 \\
& \quad + 4(m_a^2 + m_b^2 + m_c^2)u^4 - 12m_a^{2/3} m_b^{2/3} m_c^{2/3}u^4]
\end{aligned}
\tag{A.1}
$$

B Bare Mass Ratios and Related u_o Singularity Locations

Mass Ratio $\frac{m_a}{m_a} : \frac{m_b}{m_a} : \frac{m_c}{m_a}$	u_o m_a (Λ)	u (Λ^2)
$1 : .007 : .00002$	1600	74
$1 : .008 : .00003$	1400	73
$1 : .5 : .5$	.32	.064
$1 : .1 : .1$	2.7	.42
$1 : .01 : .01$	58	7.6
$1 : .001 : .001$	1300	160

Table I. Bare Mass Quark Ratios and the Related u_o Singularity. Listed here are the m_a and u solutions of the u_o singularity for a few quark bare mass ratios. In these examples $0.1\,\Lambda \stackrel{<}{\sim} m_a \stackrel{<}{\sim} 2000\Lambda$ and $0.1\,\Lambda^2 \stackrel{<}{\sim} u \stackrel{<}{\sim} 200\Lambda^2$. For a given mass ratio, u_- and u_+ generally offer m_a and u solutions similar in magnitude to those of u_o. (For the u_- solutions m_a and u are of opposite sign.) However, corresponding u_- or u_+ solutions sometimes fail to exist when $m_b \approx m_c$.

For generic mass ratios (excepting those where $m_b/m_a \sim m_c/m_a \sim \mathcal{O}(1)$), the u_- and u_+ singularities offer three additional classes of strong coupling solutions that u_o does not provide. These solutions involve fine tuning of m_a and u though. The u_+ singularity yields solutions where (i) $\mathcal{O}(m_a) \sim \frac{1+\beta}{32}\Lambda$, with $|\beta| < .01$ and $\frac{1}{256}\Lambda^2 \leq u < \frac{1}{160}\Lambda^2$, and (ii) $\mathcal{O}(m_a) \stackrel{<}{\sim} 10^{-21}\Lambda$ and $u = \frac{1+\epsilon}{256}\Lambda^2$, with $|\epsilon| < 10^{-8}$. The u_- singularity solutions produce the same mass ratios and $\mathcal{O}(m_a) \stackrel{<}{\sim} 10^{-21}\Lambda$ mass as in (ii) above, but require $\mathcal{O}(u) \stackrel{<}{\sim} 10^{-19}\Lambda^2$ instead.

The u_+ singularity also offers *extremely*–weak coupling solutions where $\mathcal{O}(m_a) \sim 10^{8 \text{ to } 12}\Lambda$ and $\mathcal{O}(u) \sim 10^{9 \text{ to } 14}\Lambda^2$. For example, the $1 : .007 : .00003$ ratio occurs at $m_a \approx 6.2 \times 10^{11}\Lambda$ and $u \approx 1.8 \times 10^{14}\Lambda^2$.

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